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INDEX MATRICES AND OLAP-CUBE PART 3: A PRESENTATION OF THE OLAP "INTERCUBE SET" AND "DATA CUBE" OPERATIONS BY INDEX MATRICES

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Abstract. In the current paper an interpretation of the OLAP cube using the apparatus of index matrices is presented. The OLAP "InterCube Set" and "Data cube" operations are defined by apparatus of index matrices. Some examples of these operations by MDX (MultiDimensional eXpressions) [23] are given in the paper.

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1. Introduction

In the context of data warehousing, the literature proposes several approaches to multidimensional modeling. Some of them have no graphical support and are aimed at establishing a formal foundation for representing cubes and hierarchies as well as an algebra for querying them [1, 12, 15, 16, 17, 21, 24, 32], while we believe that a distinguishing feature of conceptual models is that of providing a graphical support to be easily understood by both designers and users.

The concept of OLAP (online analytical processing) has been introduced by Codd in 1993 (see [13]). The aim of OLAP is to support ad-hoc querying for the business analyst. Using OLAP tools users can navigate through and analyze multidimensional data. OLAP provides technique of performing complex analysis over the information stored in a data warehouse. There are many papers describing the type, the architecture and the applications related with OLAP [2, 14, 18, 19, 20, 22, 25]. In [25] are presented the basic requirements to the OLAP Cube, which are the following:

- Core Logical Requirements of OLAP
  (1) Rich dimensional structuring with hierarchical referencing;
  (2) Efficient Specification of Dimensions and Calculations;
  (3) Flexibility;

(4) Separation of Structure and Representation;
- Core Physical Requirements
  (1) Fast Access;
  (2) Multiuser Support.

At a high level steps of designing OLAP-cube, implementing a multidimensional information, include the following [25]:
- Understanding the current and ideal data flow;
- Defining cubes;
- Defining dimensions, members, and links;
- Defining dimension levels and hierarchies;
- Defining aggregations and other formulas.

In the current paper, which is a continuation of the articles [11, 29], "InterCube Set" and "Data cube" operations over OLAP cube will be discussed. Also will be defined their index matrix interpretations and some practical examples will be considered.

The theory of Index Matrices (IMs) is introduced in [3]. Apparatus of 3D-Extended Index Matrices (3D-EIMs) is defined in [5] and further developed in [4, 26, 27, 31]. The practical examples in the paper are performed using Multidimensional expressions language (MDX). MDX provides a syntax for querying and manipulating the multidimensional data [23].

For the needs of the present research the definition of a 3D-EIM and some operations over them will be recalled in Section 2. In Sections 3 and 4 we will present the definitions of "InterCube Set" and "Data cube" operations over OLAP-cube and will be performed by the apparatus of the IMs. Some example applications of these operations, described by MDX [23], will be given.

2. Short remarks on 3D-Extended index matrix

Let us start with a definition of a 3D-EIM from [5, 10], which was extended in [27].

2.1. **Definition of 3D-EIM and some operations over them.**

2.1.1. *Definition of 3D-EIM.* The Intuitionistic Fuzzy Pair (IFP) [6, 9] is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let $\mathcal{I}$ be a fixed set of indixes,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1\leq n\leq\infty} \mathcal{I}^n.$$

Let $\mathcal{X}$ be a fixed set of some objects. In the particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, IFPs, function etc.

An "3D-Extended Index Matrix" (3D-EIM) with index sets K, L and $H (K, L, H \subset \mathcal{I}^*)$ and elements from set $\mathcal{X}$ is called the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}] = \left\{ \begin{array}{c|ccccc} h_g & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \dots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \dots & a_{k_i,l_j,h_g} & \dots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \dots & a_{k_m,l_j,h_g} & \dots & a_{k_m,l_n,h_g} \end{array} \quad | \; h_g \in H \right\}$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, $H = \{h_1, h_2, \dots, h_f\}$, and for $1 \le i \le m,\ 1 \le j \le n,\ 1 \le g \le f : a_{k_i,l_j,h_g} \in \mathcal{X}$.

Following [5, 27], let $3D - EIM_R$ be the set of all 3D-EIMs with elements being real numbers; $3D - EIM_{\{0,1\}}$ be the set of all $(0,1)$-3D-EIMs with elements being 0 or 1; $3D - EIM_P$ be the set of all 3D-EIMs with elements – predicates; $3D - EIM_{IFP}$ be the set of all 3D-EIMs with elements – IFPs and $3D - EIM_{FE}$ – the set of all 3D-EIMs with elements – 1-argument functions $\in F^{1}$.

2.1.2. *Operations with 3D-EIMs.*

- **Automatic reduction**

The definition of the operation for an 3D-EIM A following [5] is

$$@(A) = [P, Q, R, \{b_{p_r,q_s,d_e}\}],$$

where $P \subseteq K, Q \subseteq L, R \subseteq H$ are index sets with the following properties:

$$(\forall k_i \in K - P)(\forall l_j \in L)(\forall h_g \in H)(a_{k_i,l_j,h_g} = \perp)$$

$$\&\ (\forall k_i \in K)(\forall l_j \in L - Q)(\forall h_g \in H)(a_{k_i,l_j,h_g} = \perp)$$

$$\&\ (\forall k_i \in K)(\forall l_j \in L)(\forall h_g \in H - R)(a_{k_i,l_j,h_g} = \perp)$$

$$\&\ (\forall p_r = k_i \in P)(\forall q_s = l_j \in Q)(\forall d_e = h_g \in H)(b_{p_r,q_s,d_e} = a_{k_i,l_j,h_g}).$$

We can also define the following operations:

- **Automatic reduction by K**

$$@_K(A) = [P, L, H, \{b_{p_r,l_j,h_g}\}],$$

where $P \subseteq K$ are index set with the following properties:

$$(\forall k_i \in K - P)(\forall l_j \in L)(\forall h_g \in H)(a_{k_i,l_j,h_g} = \perp)$$

$$\&\ (\forall p_r = k_i \in P)(\forall l_j \in L)(\forall h_g \in H)(b_{p_r,l_j,h_g} = a_{k_i,l_j,h_g}).$$

- **Automatic reduction by L**

$$@_L(A) = [K, Q, H, \{b_{k_i,q_s,h_g}\}],$$

where $Q \subseteq L$ are index set with the following properties:

$$(\forall k_i \in K)(\forall l_j \in L - Q)(\forall h_g \in H)(a_{k_i,l_j,h_g} = \perp)$$

$$\&\ (\forall k_i \in K)(\forall q_s = l_j \in L)(\forall h_g \in H)(b_{k_i,q_s,h_g} = a_{k_i,l_j,h_g}).$$

- **Automatic reduction by H**

$$@_H(A) = [K, L, R, \{b_{k_i,l_j,d_e}\}],$$

where $R \subseteq H$ are index set with the following properties:

$$(\forall k_i \in K)(\forall l_j \in L)(\forall h_g \in H - R)(a_{k_i,l_j,h_g} = \perp)$$
$$\& \ (\forall k_i \in K)(\forall l_j \in L)(\forall d_e = h_g \in R)(b_{k_i,l_j,d_e} = a_{k_i,l_j,h_g}).$$

Analogously are the definitions of $@_{K,L}(A)$, $@_{K,H}(A)$ and $@_{L,H}(A)$.

- **Automatic zero-reduction**

In [30] is defined this operation over 3D-EIM A as follows:

$$@^0(A) = [P, Q, R, \{b_{p_r,q_s,d_e}\}]$$

where $P \subseteq K, Q \subseteq L, R \subseteq H$ are index sets with the following properties:

$$(\forall k_i \in K - P)(\forall l_j \in L)(\forall h_g \in H)(a_{k_i,l_j,h_g} = 0)$$
$$\& \ (\forall k_i \in K)(\forall l_j \in L - Q)(\forall h_g \in H)(a_{k_i,l_j,h_g} = 0)$$
$$\& \ (\forall k_i \in K)(\forall l_j \in L)(\forall h_g \in H - R)(a_{k_i,l_j,h_g} = 0)$$
$$\& \ (\forall p_r = k_i \in P)(\forall q_s = l_j \in Q)(\forall d_e = h_g \in H)(b_{p_r,q_s,d_e} = a_{k_i,l_j,h_g}).$$

- **Projection**

The operation "projection" over EIMs is defined in [5, 26, 31].

2.2. Definition of 3D-Multilayer extended index matrix and some operations over them.

2.2.1. *Definition of 3D-multilayer extended index matrix (3D-MLEIM).* Let us present the definition of 3D-MLEIM A [26, 31] with P-levels (layers) of use of dimension K, Q-levels(layers) of use of dimension L and R-levels(layers) of use of dimension H as follows:

$$A = [K, L, H, \{a_{K^{(p)}_{i,d}, L^{(q)}_{j,b}, H^{(r)}_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccc} H_g^{(R)} \in H & L_1^{(Q)} & \dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\ \hline K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_i^{(P)} & a_{K_i^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_n^{(Q)},H_g^{(R)}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}} \end{array} \right\},$$

where

$$K = \{K_1^{(P)}, K_2^{(P)}, \dots, K_i^{(P)}, \dots, K_m^{(P)}\},$$
$$K_i^{(P)} = \left\{K_{i,1}^{(P-1)}, K_{i,2}^{(P-1)}, \dots, K_{i,x}^{(P-1)}, \dots, K_{i,I}^{(P-1)}\right\} \text{ for } 1 \le i \le m$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$K_u^{(1)} = \left\{K_{u,1}^{(0)}, K_{u,2}^{(0)}, \dots, K_{u,U}^{(0)}\right\}$$

i.e. $p-$th layer of dimension K of the multilayer matrix, where $(1 \le p \le P)$, is represented by

$$K_{u_*}^{(p)} = \left\{K_{u_*,1}^{(p-1)}, K_{u_*,2}^{(p-1)}, \dots, K_{u_*,U_*}^{(p-1)}\right\} \text{ for } 1 \le p \le P$$
$$L = \{L_1^{(Q)}, L_2^{(Q)}, \dots, L_j^{(Q)}, \dots, L_n^{(Q)}\},$$

$$L_j^{(Q)} = \{L_{j,1}^{(Q-1)}, L_{j,2}^{(Q-1)}, \ldots, L_{j,y}^{(Q-1)}, \ldots, L_{j,J}^{(Q-1)}\} \text{ for } 1 \leq j \leq n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$L_v^{(1)} = \left\{L_{v,1}^{(0)}, L_{v,2}^{(0)}, \ldots, l_{v,V}^{(0)}\right\}$$

i.e. $q-$th layer of dimension Q of the multilayer matrix is represented by

$$L_{v_*}^{(q)} = \left\{L_{v_*,1}^{(q-1)}, L_{v_*,2}^{(q-1)}, \ldots, L_{v_*,V_*}^{(q-1)}\right\} \text{ for } 1 \leq q \leq Q$$
$$H = \{H_1^{(R)}, H_2^{(R)}, \ldots, H_g^{(R)}, \ldots, H_f^{(R)}\},$$
$$H_g^{(R)} = \{H_{g,1}^{(R-1)}, H_{g,2}^{(R-1)}, \ldots, H_{g,z}^{(R-1)}, \ldots, H_{g,G}^{(R-1)}\} \text{ for } 1 \leq g \leq f$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$H_w^{(1)} = \left\{H_{w,1}^{(0)}, H_{w,2}^{(0)}, \ldots, H_{w,W}^{(0)}\right\}$$

i.e. $r-$th layer of dimension H of the multilayer matrix is represented by

$$H_{w_*}^{(r)} = \left\{H_{w_*,1}^{(r-1)}, H_{w_*,2}^{(r-1)}, \ldots, H_{w_*,W_*}^{(r-1)}\right\} \text{ for } 1 \leq r \leq R$$

and $(K, L, H \subset \mathcal{I}^*)$, and for $1 \leq i \leq I,\ 1 \leq j \leq J,\ 1 \leq g \leq G,\ 1 \leq p \leq P,\ 1 \leq q \leq Q,\ 1 \leq r \leq R,\ 1 \leq d \leq I,\ 1 \leq b \leq J,\ 1 \leq c \leq G : a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} \in \mathcal{X},\ K_{i,0}^{(p)} \notin K$, $L_{j,0}^{(q)} \notin L$ and $H_{g,0}^{(r)} \notin H$.

2.2.2. *Operations with 3D-MLEIMs.*

- **Automatic reduction**

Let us have 3D-MLEIM $A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$. We will extend the definitions of the operations "Automatic reduction" over the matrix A.

Let us define "Automatic reduction" by $x-$ layer of the dimension K, $y-$ layer of the dimension L and $z-$ layer of the dimension H.

$$@_{((K,x\text{-layer}),(L,y\text{-layer}),(H,z\text{-layer}))}A = [K*, L*, H*, \{b_{K*_{i,d}^{(p*)}, L*_{j,b}^{(q*)}, H*_{g,c}^{(r*)}}\}]$$

where $K* \subseteq K, L* \subseteq L, H* \subseteq H, P* \subseteq P, Q* \subseteq Q, R* \subseteq R$ are index sets with the following properties:

$$(\forall\ K_{i,d}^{(p)} \in \{(K_i^{(P)} - K*_i^{(P*)}), x\text{-layer}\}), (\forall\ L_{j,b}^{(q)} \in \{L_j^{(Q)}, q\text{-layer}\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{H_g^{(Q)}, r\text{-layer})\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall\ K_{i,d}^{(p)} \in \{K_i^{(P)}, p\text{-layer}\})(\forall\ L_{j,b}^{(q)} \in \{(L_j^{(Q)} - L*_j^{(Q*)}), y\text{-layer}\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{H_g^{(Q)}, r\text{-layer})\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall\ K_{i,d}^{(p)} \in \{K_i^{(P)}, p\text{-layer}\})(\forall\ L_{j,b}^{(q)} \in \{L_j^{(Q)}, q\text{-layer}\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{(H_g^{(Q)} - H*_g^{(Q*)}), z\text{-layer})\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall K*_{i,d}^{(p*)} = K_{i,d}^{(p)} \in K*)(\forall L*_{j,b}^{(q*)} = L_{j,b}^{(q)} \in L*)(\forall H*_{g,c}^{(r*)} = H_{g,c}^{(r)} \in H*)$$
$$(b_{K*_{i,d}^{(p*)}, L*_{j,b}^{(q*)}, H*_{g,c}^{(r)}} = a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r*)}})$$

and

$$K_i^{(P)} \subset K, K*_i^{(P)} \subset K*, 1 \leq \{p, x\} \leq P, 1 \leq p* \leq P*, L_j^{(Q)} \subset L, L*_j^{(Q)} \subset L*,$$
$$1 \leq \{q, y\} \leq Q, 1 \leq q* \leq Q*, H_g^{(R)} \subset H, H*_g^{(R)} \subset H*, 1 \leq \{r, z\} \leq R, 1 \leq r* \leq R*.$$

Analogously, we can define the other operations of "Automatic reduction" as:

$$@_{((K,x\text{-layer}),L,H))}A, @_{(K,(L,y\text{-layer}),H)}A, @_{(K,L,(H,z\text{-layer}))}A,$$
$$@_{((K,x\text{-layer}),(L,y\text{-layer}),H)}A, @_{((K,x\text{-layer}),L,(H,z\text{-layer}))}A$$

and

$$@_{(K,(L,y\text{-layer}),(H,z\text{-layer}))}A.$$

Let us generalize these operations and define following operations of "Automatic reduction" by some layers of the dimension K, by layers of the dimension L and other layers of the dimension H.

$$@_{(K,X),(L,Y),(H,Z)}A = [K*, L*, H*, \{b_{K*_{i,d}^{(p*)}, L*_{j,b}^{(q*)}, H*_{g,c}^{(r*)}}\}]$$

where $K* \subseteq K, L* \subseteq L, H* \subseteq H$, $X \subseteq P$, $Y \subseteq Q$ and $Z \subseteq R$ are index sets with the following properties:

$$(\forall\ K_{i,d}^{(p)} \in \{(K_i^{(P)} - K*_i^{(P)}), x \in X\}), (\forall\ L_{j,b}^{(q)} \in \{L_j^{(Q)}, q\text{-layer}\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{H_g^{(Q)}, r\text{-layer})\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall\ K_{i,d}^{(p)} \in \{K_i^{(P)}, p\text{-layer}\})(\forall\ L_{j,b}^{(q)} \in \{(L_j^{(Q)} - L*_j^{(Q)}), y \in Y\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{H_g^{(Q)}, r\text{-layer})\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall\ K_{i,d}^{(p)} \in \{K_i^{(P)}, p\text{-layer}\})(\forall\ L_{j,b}^{(q)} \in \{L_j^{(Q)}, q\text{-layer}\})$$
$$(\forall\ H_{g,c}^{(r)} \in \{(H_g^{(Q)} - H*_g^{(Q)}), z \in Z)\})(a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = \perp)$$
$$\&\ (\forall K*_{i,d}^{(p*)} = K_{i,d}^{(p)} \in K*)(\forall L*_{j,b}^{(q*)} = L_{j,b}^{(q)} \in L*)(\forall H*_{g,c}^{(r*)} = H_{g,c}^{(r)} \in H*)$$
$$(b_{K*_{i,d}^{(p*)}, L*_{j,b}^{(q*)}, H*_{g,c}^{(r*)}} = a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}})$$

and

$$K_i^{(P)} \subset K, K*_i^{(P)} \subset K*, 1 \le \{p, x\} \le P, 1 \le p* \le P*, L_j^{(Q)} \subset L, L*_j^{(Q)} \subset L*,$$
$$1 \le \{q, y\} \le Q, 1 \le q* \le Q*, H_g^{(R)} \subset H, H*_g^{(R)} \subset H*, 1 \le \{r, z\} \le R, 1 \le r* \le R*.$$

If $X = P, Y = Q, Z = R$, then this operation is denoted by:

$$@A = [K*, L*, H*, \{b_{K*_{i,d}^{(p*)}, L*_{j,b}^{(q*)}, H*_{g,c}^{(r*)}}\}]$$

and it checks the matrix for empty layers of all three dimensions K, L and H.

- **Automatic zero-reduction**

Definition of operations $@^0_{((K,x\text{-layer}),(L,y\text{-layer}),(H,z\text{-layer}))}A$, $@^0_{((K,x\text{-layer}),L,H))}A$, $@^0_{(K,(L,y\text{-layer}),H)}A$, $@^0_{(K,L,(H,z\text{-layer}))}A$, $@^0_{((K,x\text{-layer}),(L,y\text{-layer}),H)}A$, $@^0_{((K,x\text{-layer}),L,(H,z\text{-layer}))}A$ $@^0_{(K,(L,y\text{-layer}),(H,z\text{-layer}))}A$, $@_{(K,X),(L,Y),(H,Z)}A$ and $@^0(A)$ over 3D-MLEIM A is analogous to operation " Automatic reduction".

In [29, 11, 31] operations "Projection" and three hierarchical operators are extended over 3D-MLEIMs. They will be used to describe the operations in OLAP-cube.

3. An implementation of the OLAP operation "Data cube" by index matrices

In the current Section is presented an implementation of the OLAP operations "Data cube" by index matrices and also are presented some examples of application of this operation. OLAP cube "Bookshops" is constructed in [29]. It contains information for the book sales in different bookshops (managed by different regional managers) in different locations. The structure of the cube "Bookshops" is visualized on Fig. 1. The fact table "Sales" and the dimensional tables BooksId, Title, Publisher, Genre, Price, BookshopsId, Bookshop Name, Regional Manager, Owner and LocationId, Town, Country are constructed. The measures are Number and Sales Count. The hierarchical structures of the dimensions are presented in [29].

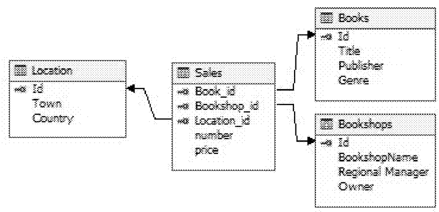

Fig. 1. Star schema "Bookshops"

In terms of 3D-EIMs the above example of OLAP-cube becomes the following: let us create matrix $A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$ (3D-MLEIM with P-levels (layers) of use of a dimension K, Q-levels (layers) of use of a dimension L and R-levels (layers) of use of a dimension H with structure, defined in [29].

3.1. **Definition.**
The "Data cube" operation is a multidimensional generalization of a cross-tab. This operation summarizes the values by rows and by columns in every step of the implementation of the operation "Roll-up" or recovers them when "Drill-down" is applied over the cube (Fig. 2). Generally, the option for aggregation with "Roll-up" operation is used. The "Data cube" operation will be presented using hierarchical operators over 3D-MLEIM [29] also.

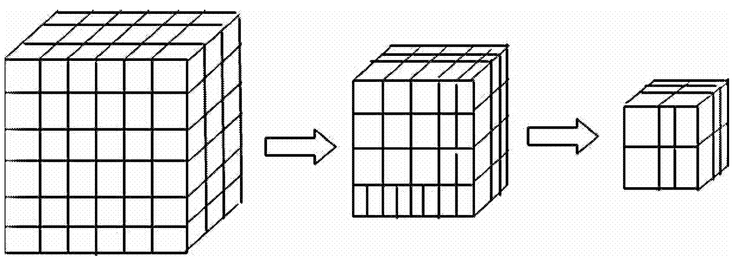

Fig. 2. Operation "Data cube"

3.2. **Presentation of the operation "Data cube" by IMs.**
In the case of 3D-EIM A: Let be given 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$. We will use the following aggregation operations, defined in [26, 28], to represent the operation "Data cube" in terms of index matrices:

$(\circ) - \alpha_{(K,K_*)}$-aggregation; $(\circ) - \alpha_{(L,L_*)}$-aggregation; $(\circ) - \alpha_{(H,H_*)}$-aggregation.

The operation "Data cube" can be represented as follows:

$$\alpha_{(K,K_*,\circ)}(A, W_*) \oplus_{(\vee)} \alpha_{(L,L_*,\circ)}(A, U_*) \oplus_{(\vee)} \alpha_{(H,H_*,\circ)}(A, V_*)$$
$$\oplus_{\vee}\alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\circ)}(A, W_*, U_*) \oplus_{(\vee)} \alpha_{(\langle K,K_*\rangle,\langle H,H_*\rangle,\circ)}(A, W_*, V_*)$$
$$\oplus_{\vee}\alpha_{(\langle L,L_*\rangle,\langle H,H_*\rangle,\circ)}(A, U_*, V_*) \oplus_{\vee} \alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\langle H,H_*\rangle,\circ)}(A, W_*, U_*, V_*)$$

where
$K_* = \{K_{w_1}, \ldots, K_{w_x}, \ldots, K_{w_W}\} \subseteq K$, $W_* = \{K_{w_1,0}, \ldots, K_{w_x,0}, \ldots, K_{w_t,0}\} \not\subset K$; $L_* = \{L_{u_1}, \ldots, L_{u_y}, \ldots, L_{u_U}\} \subseteq L$, $U_* = \{L_{u_1,0}, \ldots, L_{u_y,0}, \ldots, L_{u_U,0}\} \not\subset L$; $H_* = \{H_{v_1}, \ldots, H_{v_z}, \ldots, H_{v_V}\} \subseteq H$, $V_* = \{H_{v_1,0}, \ldots, H_{v_z,0}, \ldots, H_{v_V,0}\} \not\subset H$
or

$$\alpha_{(K,K_*,\circ)}(A, W_*) \oplus_{(\vee)} \alpha_{(L,L_*,\circ)}(A, U_*) \oplus_{(\vee)} \alpha_{(H,H_*,\circ)}(A, V_*),$$

where
$K_* = \{K_{w_1}, \ldots, K_{w_x}, \ldots, K_{w_W}\} \subseteq K$, $W_* = \{K_{w_1,0}, \ldots, K_{w_x,0}, \ldots, K_{w_W,0}\} \not\subset K$ $L_* = \{L_{u_1}, \ldots, L_{u_y}, \ldots, L_{u_U}\} \subseteq L$, $U_* = \{L_{u_1,0}, \ldots, L_{u_y,0}, \ldots, L_{u_U,0}\} \not\subset L$, $H_* = \{H_{v_1}, \ldots, H_{v_z}, \ldots, H_{v_V}\} \subseteq H$, $V_* = \{H_{v_1,0}, \ldots, H_{v_y,0}, \ldots, H_{v_V,0}\} \not\subset H$.

In the case of 3D-MLEIM $A = [K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]$*:*
Operation "Data cube" is expressed by an element of set:

$$\{(\circ) - \alpha_{(K,K_*,\text{ p-layer})} - \text{ aggregation}, (\circ) - \alpha_{(L,L_*,\text{ q-layer})} - \text{ aggregation},$$
$$(\circ) - \alpha_{(H,H_*,\text{ r-layer})} - \text{ aggregation}, (\circ) - \alpha_{(K,K_*,P_*)}, (\circ) - \alpha_{(L,L_*,Q_*)}$$
$$(\circ) - \alpha_{(H,H_*,R_*)}\}.$$

Summarized the operation "Data cube" can be represented as follows:

$$\alpha_{(K,K_*,\text{ p-layer},\circ)}(A, W_*) \oplus_{(\vee)} \alpha_{(L,L_*,\text{ q-layer},\circ)}(A, U_*)$$
$$\oplus_{(\vee)}\alpha_{(H,H_*,\text{ r-layer},\circ)}(A, V_*) \oplus_{(\vee)} \alpha_{(\langle K,K_i^{(P)},\text{ p-layer }\rangle,\langle L,L_j^{(Q)},\text{ q-layer }\rangle,\circ)}(A, W_*, U_*)$$
$$\oplus_{(\vee)}\alpha_{(\langle K,K_i^{(P)},\text{ p-layer }\rangle,\langle H,H_g^{(R)},\text{ r-layer }\rangle,\circ)}(A, W_*, V_*)$$
$$\oplus_{(\vee)}\alpha_{(\langle L,L_j^{(Q)},\text{ q-layer }\rangle,\langle H,H_g^{(R)},\text{ r-layer }\rangle,\circ)}(A, U_*, V_*)$$
$$\oplus_{(\vee)}\alpha_{(\langle K,K_i^{(P)},\text{ p-layer }\rangle,\langle L,L_j^{(Q)},\text{ q-layer }\rangle,(\langle H,H_g^{(R)},\text{ r-layer }\rangle,\circ)}(A, W_*, U_*, V_*)$$

where

$$K_* = \{K^{(P)}_{w_1}, \ldots, K^{(P)}_{w_x}, \ldots, K^{(P)}_{w_W}\} \subseteq K,$$
$$W_* = \{K^{(p)}_{w_1,0}, \ldots, K^{(p)}_{w_x,0}, \ldots, K^{(p)}_{w_t,0}\} \not\subset K, \text{ for } 1 \le p \le P;$$
$$L_* = \{L^{(Q)}_{u_1}, \ldots, L^{(Q)}_{u_y}, \ldots, L^{(P)}_{u_U}\} \subseteq L,$$
$$U_* = \{L^{(q)}_{u_1,0}, \ldots, L^{(q)}_{u_y,0}, \ldots, L^{(q)}_{u_U,0}\} \not\subset L \text{ for } 1 \le q \le Q;$$
$$H_* = \{H^{(R)}_{v_1}, \ldots, H^{(R)}_{v_z}, \ldots, H^{(R)}_{v_V}\} \subseteq H,$$
$$V_* = \{H^{(r)}_{v_1,0}, \ldots, H^{(r)}_{v_z,0}, \ldots, H^{(r)}_{v_V,0}\} \not\subset H \text{ for } 1 \le r \le R;$$

or

$$\alpha_{(K,K_*,P_*,\circ)}(A, W_*) \oplus_{(\vee)} \alpha_{(L,L_*,Q_*,\circ)}(A, U_*) \oplus_{(\vee)} \alpha_{(H,H_*,R_*,\circ)}(A, V_*)$$

$$\oplus_{(\vee)}\alpha_{(K,K_*,P_*,\circ),(L,L_*,Q_*,\circ)}(A,W_*,U_*)$$
$$\oplus_{(\vee)}\alpha_{(K,K_*,P_*,\circ),(H,H_*,R_*,\circ)}(A,W_*,V_*)$$
$$\oplus_{(\vee)}\alpha_{(L,L_*,Q_*,\circ),(H,H_*,R_*,\circ)}(A,U_*,V_*)$$
$$\oplus_{(\vee)}\alpha_{(K,K_*,P_*,\circ),(L,L_*,Q_*,\circ),(H,H_*,R_*,\circ)}(A,W_*,U_*,V_*)$$

where
$K_* = \{K^{(P)}_{w_1},\dots,K^{(P)}_{w_x},\dots,K^{(P)}_{w_W}\} \subseteq K$, $W_* = \{K^{(p_1)}_{w_1,0},\dots,K^{(p_x)}_{w_x,0},\dots,K^{(p_W)}_{w_W,0}\} \not\subset K$, $P_* = \{p_1,\dots,p_x,\dots,p_W\}$, where $p_x \in \{1,\dots,P\}$ for $1 \le x \le W'$ $L_* = \{L^{(Q)}_{u_1},\dots,L^{(Q)}_{u_y},\dots,L^{(Q)}_{u_U}\} \subseteq L$, $U_* = \{L^{(q_1)}_{u_1,0},\dots,L^{(q_y)}_{u_y,0},\dots,L^{(q_U)}_{u_U,0}\} \not\subset L$, $Q_* = \{q_1,\dots,q_y,\dots,q_Q\}$, where $q_y \in \{1,\dots,Q\}$ for $1 \le y \le U$, $H_* = \{H^{(R)}_{v_1},\dots,H^{(R)}_{v_z},\dots,H^{(R)}_{v_V}\} \subseteq H$, $V_* = \{H^{(r_1)}_{v_1,0},\dots,H^{(r_z)}_{v_y,0},\dots,H^{(r_V)}_{v_V 0}\} \not\subset H$, $R_* = \{r_1,\dots,r_z,\dots,r_V\}$, where $r_z \in \{1,\dots,R\}$ for $1 \le z \le V$.

When “Drill-down” (see [29]) operation is performed using “Data cube” operator it is necessary to use hierarchical operators to navigate in the different levels of the cube [29]:

$$A^*|(a_{K^{(P)}_{i_a},L^{(Q)}_{j_b},H^{(R)}_{gc}};(p,q,r)) \text{ or } A^*|(p,q,r),$$

where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$.

3.3. **Representation of the operation “Data cube” between 2 or more 3D-EIMs by index matrices.**

Let us be given two 3D-EIMs

$$A_1 = [K_1, L_1, H_1, \{a_{k_{i,1},l_{j,1},h_{g,1}}\}] \text{ and } A_2 = [K_2, L_2, H_2, \{a_{k_{i,2},l_{j,2},h_{g,2}}\}].$$

The representation of the operation “Data cube” in terms of index matrices is realized in the following way:

$$\alpha_{(K_1,K_{1*},\circ)}(A_1,W_{1*}) \oplus_{(\vee)} \alpha_{(L_1,L_{1*},\circ)}(A_1,U_{1*}) \oplus_{(\vee)} \alpha_{(H_1,H_{1*},\circ)}(A_1,V_{1*})$$
$$\oplus_{\vee}\alpha_{(\langle K_1,K_{1*}\rangle,\langle L_1,L_{1*}\rangle,\circ)}(A_1,W_{1*},U_{1*}) \oplus_{(\vee)} \alpha_{(\langle K_1,K_{1*}\rangle,\langle H_1,H_{1*}\rangle,\circ)}(A_1,W_{1*},V_{1*})$$
$$\oplus_{\vee}\alpha_{(\langle L_1,L_{1*}\rangle,\langle H_1,H_{1*}\rangle,\circ)}(A_1,U_{1*},V_{1*})\oplus_{\vee}\alpha_{(\langle K_1,K_{1*}\rangle,\langle L_1,L_{1*}\rangle,\langle H_1,H_{1*}\rangle,\circ)}(A_1,W_{1*},U_{1*},V_{1*})$$
$$\oplus_{\vee}\alpha_{(K_2,K_{2*},\circ)}(A_2,W_{2*}) \oplus_{(\vee)} \alpha_{(L_2,L_{2*},\circ)}(A_2,U_{2*}) \oplus_{(\vee)} \alpha_{(H_2,H_{2*},\circ)}(A_2,V_{2*})$$
$$\oplus_{\vee}\alpha_{(\langle K_2,K_{2*}\rangle,\langle L_2,L_{2*}\rangle,\circ)}(A_2,W_{2*},U_{2*}) \oplus_{(\vee)} \alpha_{(\langle K_2,K_{2*}\rangle,\langle H_2,H_{2*}\rangle,\circ)}(A_2,W_{2*},V_{2*})$$
$$\oplus_{\vee}\alpha_{(\langle L_2,L_{2*}\rangle,\langle H_2,H_{2*}\rangle,\circ)}(A_2,U_{2*},V_{2*})\oplus_{\vee}\alpha_{(\langle K_2,K_{2*}\rangle,\langle L_2,L_{2*}\rangle,\langle H_2,H_{2*}\rangle,\circ)}(A_2,W_{2*},U_{2*},V_{2*}),$$

where

$$K_{s*} = \{K^{(P)}_{w_s,1},\dots,K^{(P)}_{w_s,x},\dots,K^{(P)}_{w_s,W_s}\} \subseteq K_s,$$
$$W_{s*} = \{K^{(p_1)}_{w_s,1,0},\dots,K^{(p_x)}_{w_s,x,0},\dots,K^{(p_W)}_{w_s,W_s,0}\} \not\subset K,$$
$$P_{s*} = \{p_{s,1},\dots,p_{s,x},\dots,p_{s,W_s}\}, \text{ where } p_{s,x} \in \{1,\dots,P_s\} \text{ for } 1 \le x \le W_s,$$
$$L_{s*} = \{L^{(Q)}_{u_s,1},\dots,L^{(Q)}_{u_s,y},\dots,L^{(Q)}_{u_s,U_s}\} \subseteq L_s,$$
$$U_{s*} = \{L^{(q_1)}_{u_s,1,0},\dots,L^{(q_y)}_{u_s,y,0},\dots,L^{(q_U)}_{u_s,U_s,0}\} \not\subset L,$$
$$Q_{s*} = \{q_{s,1},\dots,q_{s,y},\dots,q_{s,Q}\}, \text{ where } q_{s,y} \in \{1,\dots,Q_s\} \text{ for } 1 \le y \le U_s,$$
$$H_{s*} = \{H^{(R)}_{v_s,1},\dots,H^{(R)}_{v_s,z},\dots,H^{(R)}_{v_s,V_s}\} \subseteq H_s,$$
$$V_{s*} = \{H^{(r_1)}_{v_s,1,0},\dots,H^{(r_z)}_{v_s,y,0},\dots,H^{(r_V)}_{v_s,V_{s0}}\} \not\subset H,$$
$$R_{s*} = \{r_{s,1},\dots,r_{s,z},\dots,r_{s,V}\}, \text{ where } r_{s,z} \in \{1,\dots,R\} \text{ for } 1 \le z \le V, \text{ for } s = 1,2.$$

When performing indexed matrix operations, there is no requirement that their corresponding dimensions be equal to that required for standard matrix operations. The operation "Data cube" can be extended by applying to more than two cubes analogously.

Prior to using this operation, level operators [30], which set requirements for the degrees of membership and non-membership of the matrix elements, can be applied to index matrices representing OLAP-cubes. This will only process data that meets certain criteria.

3.4. **Examples for Operation "Data cube".** The following MDX queries return the highest and the lowest levels of the dimensions Books and Bookshops with the Crosstab summarization. The results are presented using the software Excel with Analysis Services and Microsoft SQL Server Management Studio.

- *MDX query1:* Operator "Data cube" in the highest level of the dimensions, performed in Microsoft SQL Server Management Studio. The MDX query is presented below.

```
SELECT NON EMPTY Hierarchize(DrilldownMember
({{{DrilldownLevel({[Bookshops].[Hierarchy].[All]})}}},
{[Bookshops].[Hierarchy].[Regional Manager].&[Richard Gray]}))
DIMENSION PROPERTIES
PARENT_UNIQUE_NAME,[Bookshops].[Hierarchy].
[Regional Manager].[Owner],
[Bookshops].[Hierarchy].[Bookshop Name].[Regional Manager]
ON COLUMNS,
NON EMPTY Hierarchize({DrilldownLevel({[Books].[Hierarchy].[All]})})
DIMENSION PROPERTIES PARENT_UNIQUE_NAME ON ROWS
FROM [Bookshops2]
WHERE ([Measures].[Sales Count]).
```

  Result: The result of the query is $2D$-table which visualizes operation "Crosstab" (Fig. 3) in the highest level of the dimensions, performed in Microsoft SQL Server Management Studio.

	All	Olivia Gomez	Stamen Dimitrov	Valeri Rodev
All	102	21	61	20
Children Books	21	4	13	4
Computer Books	71	15	42	14
Cooking Books	10	2	6	2

FIG. 3. Operation "Crosstab" in the highest level of the dimensions

- *MDX query2:* Operator "Data cube" in the lower level of the dimensions, performed in Microsoft SQL Server Management Studio. The MDX query is presented below.

```
SELECT NON EMPTY [Bookshops].[Hierarchy].MEMBERS ON COLUMNS,
NON EMPTY [Books].[Hierarchy].MEMBERS ON ROWS
```

FROM [Bookshops2]
WHERE ([Measures].[Sales Count]).

Result: The result of the query is $2D$-table (Fig. 4) which visualizes operation “Crosstab” in the lower level of the dimensions, performed in Microsoft SQL Server Management Studio.

	All	Olivia Gomez	Richard Gray	Roses 1	Stamen Dimitrov	Ivan Ivanov	Helicon 3	Helikon 1	Helikon 2	Valeri Rodev	Valeria Dimitrova	Penguins 1
All	102	21	21	21	61	61	20	20	21	20	20	20
Children Books	21	4	4	4	13	13	4	4	5	4	4	4
HarperCollins	11	2	2	2	7	7	2	2	3	2	2	2
Circles	6	1	1	1	4	4	1	1	2	1	1	1
Mulan	5	1	1	1	3	3	1	1	1	1	1	1
Orion	10	2	2	2	6	6	2	2	2	2	2	2
Bears Don't Read	5	1	1	1	3	3	1	1	1	1	1	1
Sparky	5	1	1	1	3	3	1	1	1	1	1	1
Computer Books	71	15	15	15	42	42	14	14	14	14	14	14
AlexSoft	5	1	1	1	3	3	1	1	1	1	1	1
Microsoft PowerPoint 2013 - Step by Step	5	1	1	1	3	3	1	1	1	1	1	1
Assenevci	5	1	1	1	3	3	1	1	1	1	1	1
Python-practical programming	5	1	1	1	3	3	1	1	1	1	1	1
Manning Publications	10	2	2	2	6	6	2	2	2	2	2	2
R in Action	5	1	1	1	3	3	1	1	1	1	1	1
The C# Programming Yellow Book	5	1	1	1	3	3	1	1	1	1	1	1
Microsoft Press	15	3	3	3	9	9	3	3	3	3	3	3
Business Analysis	5	1	1	1	3	3	1	1	1	1	1	1
Introducing Microsoft SQL server 2014	5	1	1	1	3	3	1	1	1	1	1	1
Programming Microsoft SQL Server 2012	5	1	1	1	3	3	1	1	1	1	1	1
O'Reily	31	7	7	7	18	18	6	6	6	6	6	6
C# 5.0 in a Nutshell: The Definitive Refer...	5	1	1	1	3	3	1	1	1	1	1	1
Discovering Statistics Using R	5	1	1	1	3	3	1	1	1	1	1	1
Hadoop: The Definitive Guide	6	2	2	2	3	3	1	1	1	1	1	1
JavaScript: The Definitive Guide	5	1	1	1	3	3	1	1	1	1	1	1

FIG. 4. Operation “Crosstab” in the lower level of the dimensions

- *MDX query3:* Operator “Data cube” in the highest level of the dimensions, performed in Microsoft Excel with Analysis Services. The MDX query is presented below.
SELECT NON EMPTY Hierarchize(DrilldownMember
({{{DrilldownLevel({[Bookshops].[Hierarchy].[All]})}}},
{[Bookshops].[Hierarchy].[Regional Manager].&[Richard Gray]}))
DIMENSION PROPERTIES PARENT_UNIQUE_NAME,
[Bookshops].[Hierarchy].[Regional Manager].[Owner],
[Bookshops].[Hierarchy].[Bookshop Name].[Regional Manager]
ON COLUMNS,
NON EMPTY Hierarchize({DrilldownLevel({[Books].[Hierarchy].[All]})})
DIMENSION PROPERTIES PARENT_UNIQUE_NAME
ON ROWS FROM [Bookshops2]
WHERE ([Measures].[Sales Count]).

Result: The result of the query (Fig. 5) visualizes operation “Crosstab” in the highest level of the dimensions, performed in Microsoft Excel with Analysis Services (OLAP add-in).

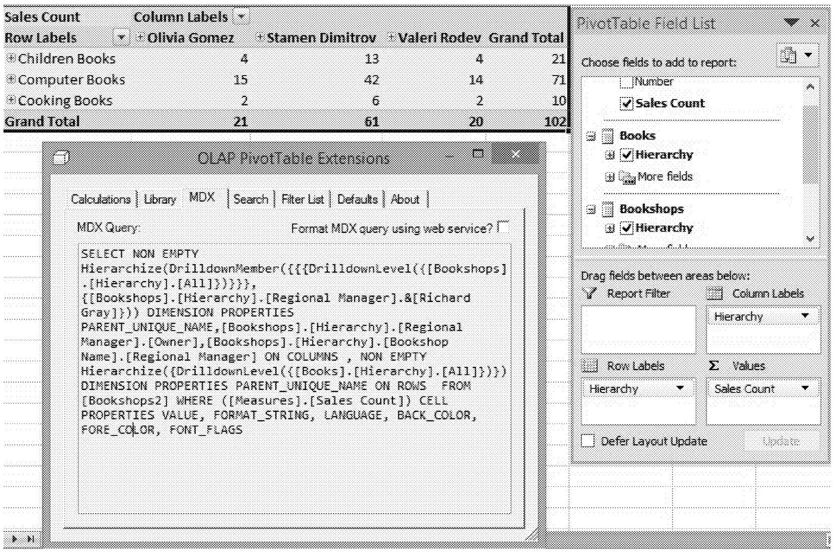

FIG. 5. Operation "Crosstab" in the highest level of the dimensions

- *MDX query4:* Operator "Data cube" in the lowest level of the dimensions, performed in Microsoft Excel with Analysis Services (OLAP add-in). The MDX query is presented below.
  SELECT NON EMPTY Hierarchize(DrilldownMember
  ({{DrilldownMember ({{DrilldownLevel({[Bookshops].[Hierarchy].[All]})}},
  {[Bookshops].[Hierarchy].[Owner].&[Olivia Gomez],
  [Bookshops].[Hierarchy].[Owner].&[Stamen Dimitrov],
  [Bookshops].[Hierarchy].[Owner].&[Valeri Rodev]})}},
  {[Bookshops].[Hierarchy].[Regional Manager].&[Richard Gray],
  [Bookshops].[Hierarchy].[Regional Manager].&[Ivan Ivanov],
  [Bookshops].[Hierarchy].[Regional Manager].&[Valeria Dimitrova]}))
  DIMENSION PROPERTIES PARENT_UNIQUE_NAME,
  [Bookshops].[Hierarchy].[Regional Manager].[Owner],
  [Bookshops].[Hierarchy].[Bookshop Name].[Regional Manager]
  ON COLUMNS,
  NON EMPTY Hierarchize(DrilldownMember({{DrilldownMember
  ({{DrilldownLevel({[Books].[Hierarchy].[All]})}},
  {[Books].[Hierarchy].[Genre].&[Children Books],
  [Books].[Hierarchy].[Genre].&[Computer Books],
  [Books].[Hierarchy].[Genre].&[Cooking Books]})}},
  {[Books].[Hierarchy].[Publisher].&[HarperCollins],
  [Books].[Hierarchy].[Publisher].&[Orion],[Books].[Hierarchy].[Publisher].
  &[AlexSoft],[Books].[Hierarchy].[Publisher].&[Assenevci],
  [Books].[Hierarchy].[Publisher].&[Manning Publications],
  [Books].[Hierarchy].[Publisher].&[Microsoft Press],
  [Books].[Hierarchy].[Publisher].&[O'Reilly],
  [Books].[Hierarchy].[Publisher].&[Springer],

[Books].[Hierarchy].[Publisher].&[Ten Speed Press]}))
DIMENSION PROPERTIES PARENT_UNIQUE_NAME,
[Books].[Hierarchy].[Publisher].[Genre],[Books].[Hierarchy].[Title].
[Publisher] ON ROWS
FROM [Bookshops2] WHERE ([Measures].[Sales Count]).

Result: The result of the query (Fig. 6) is $2D$-table which visualizes operation "Crosstab" in the lowest level of the dimensions performed in Microsoft Excel with Analysis Services (OLAP add-in).

Sales Count	Column Labels											
	Olivia Gomez		Olivia Gomez Total	Stamen Dimitrov				Stamen Dimitrov Total	Valeri Rodev		Valeri Rodev Total	Grand Total
	Richard Gray	Richard Gray Total		Ivan Ivanov			Ivan Ivanov Total		Valeria Dimitrova	Valeria Dimitrova Total		
Row Labels	Roses 1			Helicon 3	Helicon 1	Helicon 2			Penguins 1			
Children Books	4	4	4	4	4	5	13	13	4	4	4	21
HarperCollins	2	2	2	2	2	3	7	7	2	2	2	11
Circles	1	1	1	1	1	2	4	4	1	1	1	6
Mulan	1	1	1	1	1	1	3	3	1	1	1	5
Orion	2	2	2	2	2	2	6	6	2	2	2	10
Bears Don't Read	1	1	1	1	1	1	3	3	1	1	1	5
Sparky	1	1	1	1	1	1	3	3	1	1	1	5
Computer Books	15	15	15	14	14	14	42	42	14	14	14	71
AlexSoft	1	1	1	1	1	1	3	3	1	1	1	5
Microsoft PowerPoint 2013 - Step by Step	1	1	1	1	1	1	3	3	1	1	1	5
Assenevci	1	1	1	1	1	1	3	3	1	1	1	5
Python-practical programming	1	1	1	1	1	1	3	3	1	1	1	5
Manning Publications	2	2	2	2	2	2	6	6	2	2	2	10
R in Action	1	1	1	1	1	1	3	3	1	1	1	5
The C# Programming Yellow Book	1	1	1	1	1	1	3	3	1	1	1	5
Microsoft Press	3	3	3	3	3	3	9	9	3	3	3	15
Business Analysis	1	1	1	1	1	1	3	3	1	1	1	5
Introducing Microsoft SQL server 2014	1	1	1	1	1	1	3	3	1	1	1	5
Programming Microsoft SQL Server 2012	1	1	1	1	1	1	3	3	1	1	1	5
O'Reilly	7	7	7	6	6	6	18	18	6	6	6	31
C# 5.0 in a Nutshell: The Definitive Reference	1	1	1	1	1	1	3	3	1	1	1	5
Discovering Statistics Using R	1	1	1	1	1	1	3	3	1	1	1	5
Hadoop: The Definitive Guide	2	2	2	1	1	1	3	3	1	1	1	6
JavaScript: The Definitive Guide	1	1	1	1	1	1	3	3	1	1	1	5
JavaScript: The Good Parts	1	1	1	1	1	1	3	3	1	1	1	5
Learning Redis	1	1	1	1	1	1	3	3	1	1	1	5
Springer	1	1	1	1	1	1	3	3	1	1	1	5
Introduction in Computer Design	1	1	1	1	1	1	3	3	1	1	1	5
Cooking Books	2	2	2	2	2	2	6	6	2	2	2	10
Ten Speed Press	2	2	2	2	2	2	6	6	2	2	2	10
La Repertoire de la Cuisine	1	1	1	1	1	1	3	3	1	1	1	5
My Paris Kitchen: Recipes and Stories	1	1	1	1	1	1	3	3	1	1	1	5
Grand Total	21	21	21	20	20	21	61	61	20	20	20	102

FIG. 6. Operation "Crosstab" in the lowest level of the dimensions

4. AN IMPLEMENTATION OF THE OLAP INTERCUBE SET OPERATIONS BY INDEX MATRICES

In the current Section of the paper an implementation of the OLAP InterCube Set operations is presented. These operations apply over the dimensions of the OLAP cube.

We will use the constructed OLAP cube "Bookshops" in [29] for the research needs. Let us create matrix $A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$ (3D-MLEIM with P-levels (layers) of use of a dimension K, Q-levels (layers) of use of a dimension L and R-levels (layers) of use of a dimension H with structure, defined in [29].

We will represent the OLAP InterCube Set operations by index matrices in two cases:

- Case 1: *In the case of* 3D*-EIMs:*

  Let, there be given a 3D-EIM $A = [K, L, H, \{a_{k_i, l_j, h_g}\}]$.

  Let K_s, L_s and H_s for $s = 1, 2$ are index sets so that:

$$K_s \subseteq K \text{ and } K_s = \{K_{s,v_1}, \ldots, K_{s,v_x}, \ldots, K_{s,v_t}\},$$

$$L_s \subseteq L \text{ and } L_s = \{L_{s,u_1}, \ldots, L_{s,u_y}, \ldots, L_{s,u_b}\},$$

$$H_s \subseteq H \text{ and } H_s = \{H_{s,w_1}, \ldots, H_{s,w_z}, \ldots, H_{s,w_d}\}.$$

- Case 2: *In the case of 3D-MLEIM* $A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$*:*

  Let, there be given index sets for $s = 1, 2$

$$K_s \subseteq K \text{ and } K_s = \{K_{s,v_1}^{(P)}, \ldots, K_{s,v_x}^{(P)}, \ldots, K_{s,v_t}^{(P)}\},$$

$$P_s = \{p_{s,1}, \dots, p_{s,x}, \dots, p_{s,t}\}, \text{ where } 1 \le p_{s,x} \le P \text{ for } 1 \le x \le t,$$
$$L_s \subseteq L \text{ and } L_s = \{L^{(Q)}_{s,u_1}, \dots, L^{(Q)}_{s,u_y}, \dots, L^{(P)}_{s,u_b}\},$$
$$Q_s = \{q_{s,1}, \dots, q_{s,y}, \dots, q_{s,b}\}, \text{ where } 1 \le q_{s,y} \le Q \text{ for } 1 \le y \le b,$$
$$H_s \subseteq H \text{ and } H_s = \{H^{(R)}_{s,w_1}, \dots, H^{(R)}_{s,w_z}, \dots, H^{(R)}_{s,w_e}\},$$
$$R_s = \{r_{s,1}, \dots, r_{s,z}, \dots, r_{s,e}\}, \text{ where } 1 \le r_{s,z} \le R \text{ for } 1 \le z \le e.$$

Let us denote by $|X|$ the cardinality of the set X. Then $|K_s| = |P_s| = t, |L_s| = |Q_s| = b, |H_s| = |R_s| = e$.

4.1. **Operation "InterCube Difference".**

4.1.1. *Definition.* The operation "Difference" performing subtraction for the subsets of the dimensions in the OLAP cube and returns the cells that are not found in the second subset but exist in the first (Fig. 7).

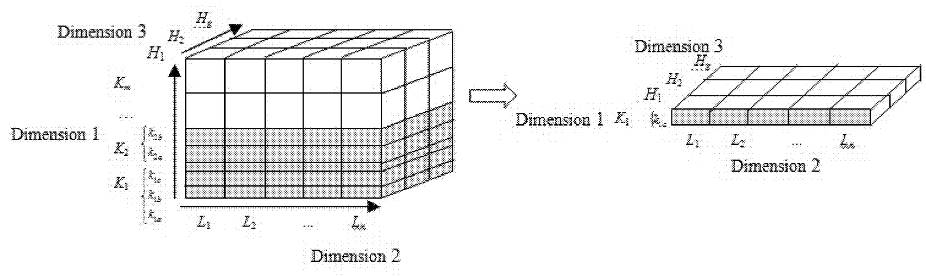

FIG. 7. Operation "Difference" by the dimension K

4.1.2. *Presentation of the operation "InterCube Difference" by the IMs. Case 1: In the case of* 3*D-EIMs:* For the presentation of the operation in terms of IMs we will use operation "Projection", which definition is given in [5, 11, 31].

Let, there be given a 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$.

Then operation "Difference" is expressed by:

$$pr_{(K_1-K_2),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L_1-L_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1-H_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1-K_2),(L_1-L_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1-K_2),L,(H_1-H_2)}A \oplus_{(\vee,\wedge)} pr_{K,(L_1-L_2),(H_1-H_2)}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1-K_2),(L_1-L_2),(H_1-H_2)}A,$$

where "-" makes sense to the standard operation "subtracting" sets.

Case 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K^{(p)}_{i,d}, L^{(q)}_{j,b}, H^{(r)}_{g,c}}\}]$*:*

The operation "Difference" is expressed by:

$$pr_{(K^{(P)}_1-K^{(P)}_2, p\text{-layer}),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L^{(Q)}_1-L^{(Q)}_2, q\text{-layer}),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H^{(R)}_1-H^{(R)}_2, r\text{-layer})}A \oplus_{(\vee,\wedge)} pr_{(K^{(P)}_1-K^{(P)}_2, p\text{-layer})(L^{(Q)}_1-L^{(Q)}_2, q\text{-layer}),H)}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K^{(P)}_1-K^{(P)}_2, p\text{-layer}),L,(H^{(R)}_1-H^{(R)}_2, r\text{-layer})}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,(L^{(Q)}_1-L^{(Q)}_2, q\text{-layer}),(H^{(R)}_1-H^{(R)}_2, r\text{-layer})}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K^{(P)}_1-K^{(P)}_2, p\text{-layer}),(L^{(Q)}_1-L^{(Q)}_2, q\text{-layer}),(H^{(R)}_1-H^{(R)}_2, r\text{-layer})}A,$$

where $1 \leq p \leq P, 1 \leq q \leq Q$ and $1 \leq r \leq R$.

The operation "InterCube Difference" can be performed succcessfully at the levels in the hierarchies as follows:

$$pr_{(K_1,P_1)-(K_2,P_2),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1)-(L_2,Q_2),H}A$$

$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1,R_1)-(H_2,R_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1)-(K_2,P_2),(L_1,Q_1)-(L_2,Q_2),H}A$$

$$\oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1)-(L_2,Q_2),(H_1,R_1)-(H_2,R_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1)-(K_2,P_2),L,(H_1,R_1)-(H_2,R_2)}A$$

$$\oplus_{(\vee,\wedge)} pr_{(K_1,P_1)-(K_2,P_2),(L_1,Q_1)-(L_2,Q_2),(H_1,R_1)-(H_2,R_2)}A.$$

4.1.3. *Examples for the operation "InterCube Difference" ("-" operator in MDX).* The operation "Difference" is performed using the function *Except*(). It returns the difference of two sets, removing the duplicate members.

- *MDX query1:* The query returns all the genres without cooking books and their sales in the countries if they have ones (Fig. 8).

  SELECT [Location].[HierarchyLocation].Children ON COLUMNS,
  NON EMPTY Except([Books].[HierarchyBooks].[All].Children,
  [Books].[HierarchyBooks].[Genre].&[Cooking Books], ALL) ON ROWS
  FROM [Bookshops2]
  WHERE ([Measures].[Sales Count])

Result: The result of the MDX-query is $2D$-table which visualizes operation "Difference", performed using function Except().

	Bulgaria	England	Turkey
Children Books	13	4	4
Computer Books	42	15	14

FIG. 8. MDX query performing operation "Difference" using function Except()

4.2. **Operation "InterCube Intersection".**

4.2.1. *Definition.* The operation "Intersection" comparing the subsets of the dimensions in the OLAP cube and returns the cells that are found in the both subsets (Fig. 9).

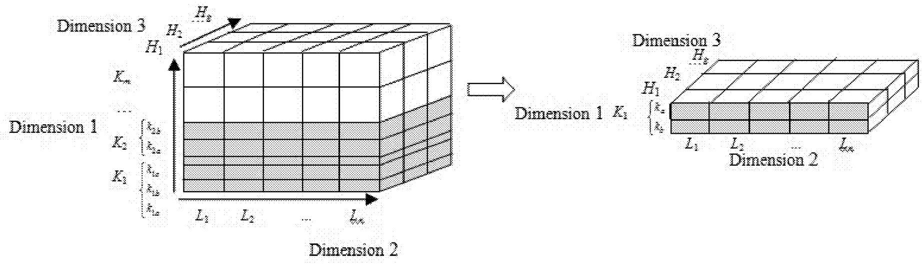

FIG. 9. Examples of the operation "InterCube Intersection" by the dimension K

4.2.2. *Presentation of the operation "InterCube Intersection" by the IMs. Case 1: In the case of* 3*D-EIMs:* The presentation of the operation in terms of the IMs is realized by operation "Projection", which definition is given in [5, 11, 31].

Let there be given a 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$.

Then operation "Intersection" is expressed by:

$$pr_{(K_1 \cap K_2),L,H} A \oplus_{(\vee,\wedge)} pr_{K,(L_1 \cap L_2),H} A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1 \cap H_2)} A \oplus_{(\vee,\wedge)} pr_{(K_1 \cap K_2),(L_1 \cap L_2),H} A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1 \cap K_2),L,(H_1 \cap H_2)} A \oplus_{(\vee,\wedge)} pr_{K,(L_1 \cap L_2),(H_1 \cap H_2)} A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1 \cap K_2),(L_1 \cap L_2),(H_1 \cap H_2)} A,$$

where "$\cap$" makes sense to the standard operation "Set intersection".

Case 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}}\}]$*:*

The operation "Intersection" is expressed by:

$$pr_{(K_1^{(P)} \cap K_2^{(P)}, p\text{-layer}),L,H} A \oplus_{(\vee,\wedge)} pr_{K,(L_1^{(Q)} \cap L_2^{(Q)}, q\text{-layer}),H} A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1^{(R)} \cap H_2^{(R)}, r\text{-layer})} A \oplus_{(\vee,\wedge)} pr_{(K_1^{(P)} \cap K_2^{(P)}, p\text{-layer}),(L_1^{(Q)} \cap L_2^{(Q)}, q\text{-layer}),H)} A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1^{(P)} \cap K_2^{(P)}, p\text{-layer}),L,(H_1^{(R)} \cap H_2^{(R)}, r\text{-layer})} A$$
$$\oplus_{(\vee,\wedge)} pr_{K,(L_1^{(Q)} \cap L_2^{(Q)}, q\text{-layer}),(H_1^{(R)} \cap H_2^{(R)}, r\text{-layer})} A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1^{(P)} \cap K_2^{(P)}, p\text{-layer}),(L_1^{(Q)} \cap L_2^{(Q)}, q\text{-layer}),(H_1^{(R)} \cap H_2^{(R)}, r\text{-layer})} A,$$

where $1 \le p \le P, 1 \le q \le Q$ and $1 \le r \le R$.

The operation "InterCube Intersection" can be performed succcessfully at the levels in the hierarchies as follows:

$$pr_{(K_1,P_1) \cap (K_2,P_2),L,H} A \oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1) \cap (L_2,Q_2),H} A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1,R_1) \cap (H_2,R_2)} A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1) \cap (K_2,P_2),(L_1,Q_1) \cap (L_2,Q_2),H} A$$
$$\oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1) \cap (L_2,Q_2),(H_1,R_1) \cap (H_2,R_2)} A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1) \cap (K_2,P_2),L,(H_1,R_1) \cap (H_2,R_2)} A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1,P_1) \cap (K_2,P_2),(L_1,Q_1) \cap (L_2,Q_2),(H_1,R_1) \cap (H_2,R_2)} A.$$

The operation can be presented as both intersections between all the attributes and chosen one/set of attributes and between two sets of attributes belonging to the selected dimension.

4.2.3. *Examples for the operation "InterCube Intersection".* The operation "Intersection" returns the intersection of two sets. By default, the function removes duplicates from both sets prior to intersecting the sets. The two sets specified must have the same dimensionality.

- *MDX query1:* The next query returns the sold books for the different genres only for the members existing in the both sets (Fig. 10).

  SELECT INTERSECT
  {[Location].[HierarchyLocation].[Town].&[Burgas],
  [Location].[HierarchyLocation].[Town].&[Plovdiv],
  [Location].[HierarchyLocation].[Town].&[Sofia]},
  {[Location].[HierarchyLocation].[Town].&[Burgas],
  [Location].[HierarchyLocation].[Town].&[Sofia],
  [Location].[HierarchyLocation].[Town].&[London]}) ON COLUMNS,
  [Books].[HierarchyBooks].[Genre] ON ROWS
  FROM [Bookshops2]
  WHERE [Measures].[Sales Count]

Result: The result of the MDX-query (Fig. 10) is $2D$-table which visualize operation "Intersection", performed using function Intersect().

	Burgas	Sofia
Children Books	4	4
Computer Books	14	14
Cooking Books	2	2

Fig. 10. MDX query performing operation "intersection" using the function Intersect()

- *MDX query2:* The query returns the count of the sold books in England (Fig. 11). This country is selected in both sets.

  SELECT INTERSECT({[Location].[Hierarchy].[Country].&[Bulgaria],
  [Location].[Hierarchy].[Country].&[England]},
  [Location].[Hierarchy].[Country].&[England], All) ON COLUMNS,
  [Measures].[Sales Count] ON ROWS
  FROM [Bookshops2]

Result: The result of the MDX-query (Fig. 11) is $2D$-table which visualizes operation "Intersection", performed using function Intersect().

	England
Sales Count	21

Fig. 11. MDX query performing operation "Intersection" using the function Intersect()

4.3. **Operation "InterCube Union".**

4.3.1. *Definition.* The operation "Union" is performed on the certain dimensional level of the hierarchy. Operation "Union" groups subsets on the level in certain dimensions (Fig. 12).

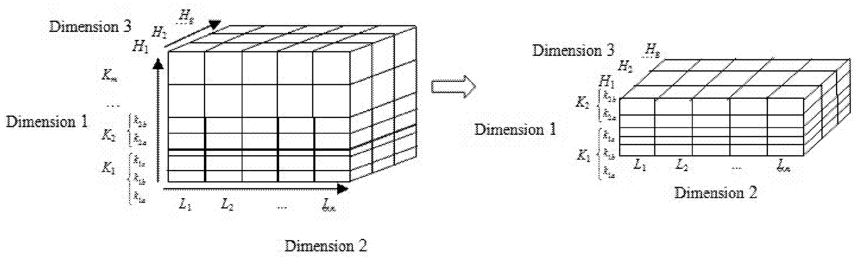

FIG. 12. Example of the operation "Union" by the dimension K

4.3.2. *Presentation of the operation "InterCube Union" by the IMs.*

Case 1: In the case of 3*D-EIMs:* The presentation of the operation in terms of the IMs is realized by operation "Projection", which definition is given in [11], is as follows:

Let there be given a 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$.

Then operation "Union" is expressed by:

$$pr_{(K_1\cup K_2),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L_1\cup L_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1\cup H_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1\cup K_2),(L_1\cup L_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1\cup K_2),L,(H_1\cup H_2)}A \oplus_{(\vee,\wedge)} pr_{K,(L_1\cup L_2),(H_1\cup H_2)}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1\cup K_2),(L_1\cup L_2),(H_1\cup H_2)}A,$$

where "$\cup$" makes sense to the standard operation "Set union".

Case 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}}\}]$*:*

The operation "Union" is expressed by:

$$pr_{(K_1^{(P)}\cup K_2^{(P)},p\text{-layer}),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L_1^{(Q)}\cup L_2^{(Q)},q\text{-layer}),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1^{(R)}\cup H_2^{(R)},r\text{-layer})}A \oplus_{(\vee,\wedge)} pr_{(K_1^{(P)}\cup K_2^{(P)},p\text{-layer}),(L_1^{(Q)}\cup L_2^{(Q)},q\text{-layer}),H)}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1^{(P)}\cup K_2^{(P)},p\text{-layer}),L,(H_1^{(R)}\cup H_2^{(R)},r\text{-layer})}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,(L_1^{(Q)}\cup L_2^{(Q)},q\text{-layer}),(H_1^{(R)}\cup H_2^{(R)},r\text{-layer})}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1^{(P)}\cup K_2^{(P)},p\text{-layer}),(L_1^{(Q)}\cup L_2^{(Q)},q\text{-layer}),(H_1^{(R)}\cup H_2^{(R)},r\text{-layer})}A,$$

where $1 \le p \le P, 1 \le q \le Q$ and $1 \le r \le R$.

The operation "InterCube Union" can be performed succcessfully at the levels in the hierarchies as follows:

$$pr_{(K_1,P_1)\cup(K_2,P_2),L,H}A \oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1)\cup(L_2,Q_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,L,(H_1,R_1)\cup(H_2,R_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1)\cup(K_2,P_2),(L_1,Q_1)\cup(L_2,Q_2),H}A$$
$$\oplus_{(\vee,\wedge)} pr_{K,(L_1,Q_1)\cup(L_2,Q_2),(H_1,R_1)\cup(H_2,R_2)}A \oplus_{(\vee,\wedge)} pr_{(K_1,P_1)\cup(K_2,P_2),L,(H_1,R_1)\cup(H_2,R_2)}A$$
$$\oplus_{(\vee,\wedge)} pr_{(K_1,P_1)\cup(K_2,P_2),(L_1,Q_1)\cup(L_2,Q_2),(H_1,R_1)\cup(H_2,R_2)}A.$$

4.3.3. *Examples for the operation "InterCube Union".* The operation "Union" returns the union of two sets by default without the duplicate members.

- *MDX query1:* The next query is the same as the previous but the titles of the selected books are grouped in the genres "Computer books" and "Cooking books" (Fig. 13).

  SELECT UNION([Location].[Hierarchy].[Country].&[Bulgaria],
  [Location].[Hierarchy].[Country].&[England]) ON COLUMNS,
  UNION([Books].[Hierarchy].[Genre].&[Computer Books],
  [Books].[Hierarchy].[Genre].&[Cooking Books]) ON ROWS
  FROM [Bookshops2]
  WHERE [Measures].[Sales Count]

Result: The result of the MDX-query is $2D$-table which visualizes the operation "Union".

	Bulgaria	England
Computer Books	42	15
Cooking Books	6	2

FIG. 13. MDX query performing the operation "Union"

- *MDX query2:* The following query returns the union of the towns in the countries Bulgaria and England. In the language MDX have not the union of the columns as the SQL language.

  SELECT UNION([Location].[HierarchyLocation].[Country].&[Bulgaria].Children,
  [Location].[HierarchyLocation].[Country].&[England].Children) ON COLUMNS,
  [Books].[HierarchyBooks].[Title] ON ROWS
  FROM [Bookshops2]

Result: The query extracts the number of the initial copies of each title in the union of the cities in the countries Bulgaria and England (Fig. 14).

	Burgas	Plovdiv	Sofia	London
Circles	4	3	2	2
Mulan	2	1	1	1
Bears Don't Read	4	3	2	1
Sparky	3	2	3	2
Microsoft PowerPoint 2013 - Step by Step	2	2	1	2
Python-practical programming	1	2	3	1
R in Action	1	2	2	3
The C# Programming Yellow Book	3	2	10	2
Business Analysis	1	2	4	2
Introducing Microsoft SQL server 2014	1	1	4	2
Programming Microsoft SQL Server 2012	1	2	9	4
C# 5.0 in a Nutshell: The Definitive Reference	2	1	7	2
Discovering Statistics Using R	2	4	6	7
Hadoop: The Definitive Guide	1	1	4	3
JavaScript: The Definitive Guide	2	1	5	3
JavaScript: The Good Parts	2	2	4	1
Learning Redis	2	2	13	1
Introduction in Computer Design	2	3	3	4
La Repertoire de la Cuisine	1	5	1	1
My Paris Kitchen: Recipes and Stories	6	1	1	2

FIG. 14. MDX query performing operation "Union"

4.4. **Operation "InterCube Cross product".**

4.4.1. *Definition.* The operation "Cross join" is performed on the certain dimensional level of the hierarchy. Operation "Union" groups subsets on the level in certain dimension like the operation "Cross join", but the difference is that the operation "Cross join" can combine subsets of different dimensions also (Fig. 15). The operation "Cross join" returns the cross product of two or more specified sets. If the sets in this operation are composed of tuples from different attribute hierarchies in the same dimension, this function will return only those tuples that actually exist. The operation can manipulate with different dimensions in the same hierarchical level on the axis also. It has several forms depending the combinations of the dimensions. One of them is:

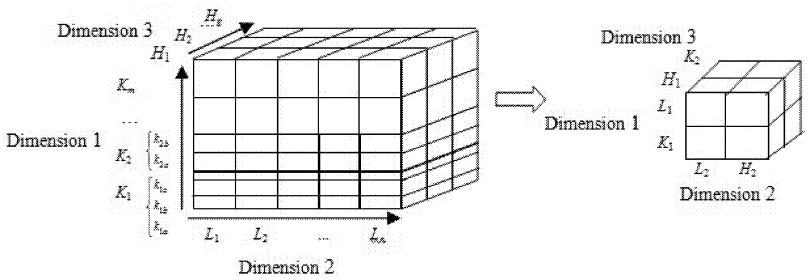

FIG. 15. Example of the operation "Cross join"

4.4.2. *Presentation of the operation "InterCube Cross Product" by the IMs.*

Case 1: In the case of 3*D-EIMs:* The presentation of the operation in terms of index matrices is realized by operation "projection", which definition is given in [5, 31] and presented in [11]:

Let there be given a 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$.

Let with T we denote $K_s \times L_s \times H_s$, $|K_s| = t$, $|L_s| = b$ and $|H_s = d$, $n = |T| = tbd$, where "$\times$" is the standard Cartesian product.

The operation "Crossjoin" is expressed by in terms of EIMs:

$$pr_{k_{s,v_1},l_{s,u_1},h_{s,w_1}} A \oplus_{(\circ)} \dots \oplus_{(\circ)} pr_{k_{s,v_1},l_{s,u_1},h_{s,w_d}} A \dots$$
$$\oplus_{(\circ)} pr_{k_{s,v_t},l_{s,u_1},h_{s,w_1}} A \oplus_{(\circ)} \dots \oplus_{(\circ)} pr_{k_{s,v_t},l_{s,u_b},h_{s,w_d}} A,$$

where ordered three indexes $\langle k_{s,v_x}, l_{s,u_y}, h_{s,w_z}\rangle \in T$. Different combinations of "cross Product" can be made according the "pivot operation". If many elements "$\perp$" or zero are obtained, then the operations of "automatic reduction" @ or $@^0$ applies to the matrix A.

Case 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]$*:*

Let with T we denote $K_s \times L_s \times H_s$, then $n = |T| = tbd$, where "$\times$" is the standard Cartesian product. The operation "product" is expressed by:

$$pr_{(K^{(P)}_{s,v_1},p\text{-layer}),(L^{(Q)}_{s,u_1},q\text{-layer}),(H^{(R)}_{s,v_1},r\text{-layer})} A \oplus_{(\circ)} \cdots$$
$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_1},p\text{-layer}),(L^{(Q)}_{s,u_1},q\text{-layer}),(H^{(R)}_{s,w_e},r\text{-layer})} A \oplus_{(\circ)} \cdots$$
$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_t},p\text{-layer}),(L^{(Q)}_{s,u_1},q\text{-layer}),(H^{(R)}_{s,w_1},r\text{-layer})} A \oplus_{(\circ)} \cdots$$
$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_t},p\text{-layer}),(L^{(Q)}_{s,u_b},q\text{-layer}),(H^{(R)}_{s,w_e},r\text{-layer})} A$$

where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$ and ordered three indexes $\langle K_{s,v_x}, L_{s,u_y}, H_{s,w_z}\rangle \in T$.

The operation "InterCube Cross Product" can be performed succcessfully at the levels in the hierarchies as follows:

$$pr_{(K^{(P)}_{s,v_1},p_{s,1}\text{-layer}),(L^{(Q)}_{s,u_1},q_{s,1}\text{-layer}),(H^{(R)}_{s,v_1},r_{s,1}\text{-layer})} A \oplus_{(\circ)} \cdots$$
$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_1},p_{s,1}\text{-layer}),(L^{(Q)}_{s,u_1},q_{s,1}\text{-layer}),(H^{(R)}_{s,w_e},r_{s,e}\text{-layer})} A \oplus_{(\circ)} \cdots$$
$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_t},p_{s,t}\text{-layer}),(L^{(Q)}_{s,u_1},q_{s,1}\text{-layer}),(H^{(R)}_{s,w_1},r_{s,1}\text{-layer})} A \oplus_{(\circ)} \cdots$$

$$\oplus_{(\circ)} pr_{(K^{(P)}_{s,v_t}, p_{s,t}\text{-layer}), (L^{(Q)}_{s,u_b}, q_{s,b}\text{-layer}), (H^{(R)}_{s,w_e}, r_{s,e}\text{-layer})} A.$$

Different combinations of "InterCube Cross Product" can be made according the pivot operation. If many elements "$\perp$" or zero are obtained, then the operations of "Automatic reduction" @ or $@^0$, defined in previous Section, applies to the matrix A.

4.4.3. *Examples for the operation "InterCube Cross Product".*

The operation "Cross join" returns the cross product of two sets.

- *MDX query1:* The following query returns "Crossjoin" of the regional managers and the cities and extract the information for the books from each genre (Fig. 16).

  SELECT CROSSJOIN([Bookshops].[HierarchyBookshops].[Regional Manager],
  [Location].[HierarchyLocation].[Town]) ON ROWS,
  NON EMPTY([Books].[HierarchyBooks].[Genre]) ON COLUMNS
  FROM [Bookshops2]
  where [Measures].[Number]

Result: The result of the MDX-query (Fig. 16) is $2D$-table which visualizes the operation "Crossjoin".

		Children Books	Computer Books	Cooking Books
Richard Gray	Burgas	(null)	(null)	(null)
Richard Gray	Plovdiv	(null)	(null)	(null)
Richard Gray	Sofia	(null)	(null)	(null)
Richard Gray	London	6	37	3
Richard Gray	Mersin	(null)	(null)	(null)
Ivan Ivanov	Burgas	13	23	7
Ivan Ivanov	Plovdiv	9	27	6
Ivan Ivanov	Sofia	8	75	2
Ivan Ivanov	London	(null)	(null)	(null)
Ivan Ivanov	Mersin	(null)	(null)	(null)
Valeria Dimitrova	Burgas	(null)	(null)	(null)
Valeria Dimitrova	Plovdiv	(null)	(null)	(null)
Valeria Dimitrova	Sofia	(null)	(null)	(null)
Valeria Dimitrova	London	(null)	(null)	(null)
Valeria Dimitrova	Mersin	6	25	7

FIG. 16. MDX query performing the operation "Crossjoin"

Obviously, in the result of the query has many empty rows. They will be removed after applying the keyword "NON EMPTY" to the query (Fig. 17).

- *MDX query2:* The following query returns "Crossjoin" after applying the keyword "NON EMPTY".

  SELECT NON EMPTY Crossjoin
  ([Bookshops].[HierarchyBookshops].[Regional Manager],

[Location].[HierarchyLocation].[Town]) ON ROWS,
NON EMPTY([Books].[HierarchyBooks].[Genre]) ON COLUMNS
FROM [Bookshops2]
where [Measures].[Number]

Result:

		Children Books	Computer Books	Cooking Books
Richard Gray	London	6	37	3
Ivan Ivanov	Burgas	13	23	7
Ivan Ivanov	Plovdiv	9	27	6
Ivan Ivanov	Sofia	8	75	2
Valeria Dimitrova	Mersin	6	25	7

FIG. 17. MDX query performing the operation "Crossjoin" with non empty

- *MDX query3:* The same query is executed by choosing the measure "Sales count" (Fig. 18):

SELECT non empty CROSSJOIN
([Bookshops].[HierarchyBookshops].[Regional Manager],
[Location].[HierarchyLocation].[Town]) ON ROWS,
NON EMPTY([Books].[HierarchyBooks].[Genre]) ON COLUMNS
FROM [Bookshops2]
WHERE [Measures].[Sales Count]

Result:

		Children Books	Computer Books	Cooking Books
Richard Gray	London	4	15	2
Ivan Ivanov	Burgas	4	14	2
Ivan Ivanov	Plovdiv	5	14	2
Ivan Ivanov	Sofia	4	14	2
Valeria Dimitrova	Mersin	4	14	2

FIG. 18. MDX query performing the operation "Crossjoin" with measure "Sales Count"

- *MDX query4:* The next query combines the operations "Union" and "Crossjoin". The result contains the count of the sold books of the genres "Computer books" and "Cooking books" in the bookshops with regional manager "Ivan Ivanov" in Bulgaria (see Fig. 19).

SELECT CROSSJOIN([Location].[Hierarchy].[Country].&[Bulgaria],
[Bookshops].[Hierarchy].[Regional Manager].&[Ivan Ivanov]) ON COLUMNS,
UNION([Books].[Hierarchy].[Genre].&[Computer Books],
[Books].[Hierarchy].[Genre].&[Cooking Books]) ON ROWS

FROM [Bookshops2]
WHERE [Measures].[Sales Count]

Result: The result of the MDX-query is $2D$-table which visualizes operation "Crossjoin".

	Bulgaria
	Ivan Ivanov
Computer Books	42
Cooking Books	6

FIG. 19. MDX query performing operation "Crossjoin" and "Union"

5. Conclusion

In the presented paper we used the index matrices as a tool for interpretation of OLAP operations "Data Cube" and "InterCube Set", which have applications in business-analysis of book sales. Operations between the 3D-EIMs - addition, subtraction - are used as a tool to create aggregated data-cubes. The outlined approach for extracting knowledge from the information stored in OLAP-cubes has the following advantages:

- Defined operations can be applied to data with explicit parameters, as well as to fuzzy or intuitionistic fuzzy parameters;
- Defined operations can be expanded to retrieve information to other types of two-dimensional or multi-dimensional data cubes [7].

Nowadays the attention is focused over the tasks for presenting the OLAP concept using 3-Dimensional and n-Dimensional IMs, and the use of these operations in the future development of the Intercriteria Decision Making method [8]. The scientific problem of OLAP-cube modeling, as well as the matrix-based interpretation of the analysis and retrieval of the information in it, has a wide practical application that makes it possible to work not only with precise parameters but also with fuzzy or intuitionistic fuzzy parameters. This paper is the third part of series of articles investigated the OLAP operations by index matrices. In the future the authors will finish the studies and some fields of application will be discussed.

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